

SOLUTION OF NONLINEAR PROBLEMS IN HEAT CONDUCTION BY
THE Z-TRANSFORM METHOD

V. L. Pokhoriler

UDC 536.24

We consider the properties of the solution of nonlinear problems in heat conduction with boundary conditions of the first kind when the Z-transform (the discrete Laplace transform) is used.

The mathematical apparatus of the Z-transform (the discrete Laplace transform) has come to be widely used for describing the dynamics of objects controlled by means of electronic computers or other digital computing systems [1]. Its use is also effective in those cases in which the monitored object is the temperature field of a body, for example of an element of a power installation [2]. The dynamic model of the object is obtained by solving the heat-conduction problem for the body in question by the Z-transform method.

The possibilities of using the Z-transform in solving nonlinear problems will be considered here by using the example of a problem in one-dimensional heat conduction in the case of a cylindrical body, the differential equation of which we shall represent in the form

$$R_{\text{h}}^2 c(\tau) \gamma(t) \frac{\partial t(\rho, \tau)}{\partial \tau} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\lambda(t) \rho \frac{\partial t(\rho, \tau)}{\partial \rho} \right]. \quad (1)$$

The boundary values will be given on the surface $\rho = k$ (an insulated surface):

$$\rho = k, \quad \frac{\partial t(k, \tau)}{\partial \rho} = 0; \quad t(\rho = k, \tau) = t(k, \tau). \quad (2)$$

This method of specifying the boundary conditions enables us to obtain a solution independent of the type of boundary conditions on the heated surface of the body, the connection to which can be established hereinafter by redefining the temperature of the insulated surface in terms of a given external-influence function (the temperature of the heated surface, the heat flux, the temperature of the heating medium for boundary conditions of the third kind) [3, 5].

The initial temperature field will be taken to be uniform:

$$t(\rho, 0) = t_i. \quad (3)$$

The case of a nonuniform initial field will be considered separately.

The hollow cylinder in this case is chosen as an example of a body with a one-dimensional temperature field. As will be shown below, the results obtained can easily be extended to an object having any geometric shape with a one-dimensional temperature field which is acted upon by an external influence through one of its surfaces and has a second surface that is thermally insulated: a plate, a sphere, etc. (A solid cylinder or sphere is a special case of a hollow cylinder or sphere.)

Our analysis showed that among the many methods for linearizing Eq. (1) described in the literature [6, 7], one of the most effective (when the Z-transform is used at the same time) is a transform defined by the relations

$$\Theta(\rho, \tau) = K \{t(\rho, \tau)\} = \int_0^{t(\rho, \tau)} \bar{\lambda}(t) dt; \quad (4a)$$

S. M. Kirov Ural Polytechnic Institute, Sverdlovsk. Translated from *Inzhenerno-Fizicheskii Zhurnal*, Vol. 51, No. 1, pp. 143-150, July, 1986. Original article submitted July 4, 1984.

$$t(\rho, \tau) = K^{-1}\{\Theta(\rho, \tau)\}. \quad (4b)$$

This transform is called Kirchhoff substitution or the Kirchhoff integral transform [7]. Its application to Eq. (1) with boundary conditions (2), (3) transforms the latter to the following form:

$$\frac{R_h^2}{a(t)} \frac{\partial \vartheta(\rho, \tau)}{\partial \tau} = \nabla^2 \vartheta(\rho, \tau); \quad (1a)$$

$$\rho = k, \quad \frac{\partial \vartheta(k, \tau)}{\partial \rho} = 0; \quad \vartheta(k, \tau) = \Theta(k, \tau) - \Theta_i; \quad (2a)$$

$$\vartheta(\rho, 0) = 0, \quad (3a)$$

where

$$\vartheta(\rho, \tau) = \Theta(\rho, \tau) - \Theta_i; \quad \Theta_i = \int_0^{t_i} \bar{\lambda}(t) dt. \quad (5)$$

The application of the Z-transform involves the use of the concept of discrete (lattice-type) functions whose values are defined for discrete values of the argument; the discretization step must necessarily be constant. The entity to which the Z-transform is applied is a difference equation (in the discretized argument), which replaces the original differential equation. The Z-transform is applicable only to linear discrete systems, in particular to those possessing the property of invariance with respect to a shift in the discrete argument [1, 8].

In solving heat-conduction problems [3, 5] with constant transfer coefficients, we obtained the linear discrete system by a simple method: replacing the continuous argument — real time τ — with a discrete argument $\tau_n = n\Delta\tau$ and replacing the first derivative of the temperature with the first difference. The difference equation so obtained is linear, and applying the Z transform to it presents no difficulties.

In the case considered here, the Kirchhoff transform carried out previously is not sufficient for obtaining a linear discrete system if we continue to use real time τ as the argument when we convert Eq. (1a) into a difference equation. The difference equation so obtained will not be invariant with respect to a shift in the discrete argument $\tau_n = n\Delta\tau$, owing to the fact that the thermal conductivity $a(t)$ is a function of temperature, i.e., ultimately a function of the argument.

The problem is solved if we take "model" time — the function

$$F_0 = \frac{1}{(R_i - R_0)^2} \int_0^\tau a(t) dt \quad (6)$$

as the discretized argument.

When we use a constant step in the discretization with respect to this argument $f = \Delta F_0 = a(t)\Delta\tau(t)/(R_i - R_0)^2$, the discrete (lattice) function of the "model" temperature $\vartheta[\rho, n]$ will be defined for the discrete values $F_{0n} = nf$, and from Eq. (1a) we obtain the linear difference equation

$$(k-1)^2 f \nabla^2 \vartheta[\rho, n] - \Delta \vartheta[\rho, n] = 0, \quad (7)$$

where $\Delta \vartheta[\rho, n] = \vartheta[\rho, n] - \vartheta[\rho, n-1]$ is the first (backward) difference of the discrete function, which is an analog of the first derivative corresponding to a continuous function.

Attention should be given to the fact that when we use a constant step $\Delta F_0 = f$ for "model" time, the discretization step with respect to real time

$$\Delta\tau(t) = f \frac{(R_i - R_0)^2}{a(t)} = \frac{a_0 \Delta\tau_0}{a(t)} = \frac{\Delta\tau_0}{a(t)} \quad (8)$$

is variable and different for different temperatures at different points of the body. For a given temperature $t(\rho)$, at the n -th calculation step F_{0n} with respect to the argument, the increment in real time is

$$\Delta\tau_n(\rho) = f \frac{(R_i - R_0)^2}{a_n^{2\nu}(\rho)} = \frac{\Delta\tau_0}{a_n^{2\nu}(\rho)}, \quad (9)$$

and to the value of the "model" time, $F_{0n} = nf$ there corresponds the "local" real time

$$\tau_n(\rho) = \tau_{n-1}(\rho) + \Delta\tau_n(\rho) = \sum_{i=1}^n \Delta\tau_i(\rho). \quad (10)$$

Application of the Z-transform by means of the relation [1, 8]

$$Z\{\tilde{\theta}[\rho, n]\} = \sum_{n=0}^{\infty} \tilde{\theta}[\rho, n] z^{-n} = \tilde{\theta}(\rho, z)$$

reduces Eq. (7) to the form

$$(k-1)^2 f \nabla^2 \tilde{\theta}(\rho, z) - (1-z^{-1}) \tilde{\theta}(\rho, z) = 0. \quad (11)$$

As was shown in [3, 5], the solution of Eq. (11), with the boundary conditions on the insulated surface ($\rho = k$)

$$\frac{d\tilde{\theta}(k, z)}{d\rho} = 0; \quad \tilde{\theta}(\rho = k, z) = \tilde{\theta}(k, z) \quad (12)$$

can be represented in the following form:

$$\tilde{\theta}(\rho, z) = \tilde{\theta}(k, z) \sum_{\nu=0}^{\infty} A_{\nu}(\rho, f) z^{-\nu} \approx \tilde{\theta}(k, z) \sum_{\nu=0}^m A_{\nu}(\rho, f) z^{-\nu}, \quad (13)$$

where

$$A_{\nu}(\rho, f) = (-1)^{\nu} \sum_{i=\nu}^m C_i^{\nu} P_i(\rho) f^{-i};$$

the C_i^{ν} are the binomial coefficients (the number of combinations), and the functions $P_i(\rho)$ satisfy the recurrent system of equations

$$\nabla^2 P_0(\rho) = 0; \quad (1-k)^2 \nabla^2 P_i(\rho) = P_{i-1}(\rho), \quad i \geq 1 \quad (14a)$$

and the boundary conditions

$$P_0(k) = 1, 0; \quad P_i(k) = 0, \quad i \geq 1; \quad \frac{dP_i(k)}{d\rho} = 0, \quad i \geq 0. \quad (14b)$$

It should be noted that the solution (13) is valid for bodies of any geometric shape which have one insulated and one heated surface, if the temperature of all the points of the heated surface is the same; only the functions $P_i(\rho)$ will be different. For bodies of canonical geometric shape — a plate, a hollow or solid cylinder, a ball or a sphere, a rod, etc. — the expressions defining the function functions $P_i(\rho)$ are determined analytically [4, 9].

The solution for mappings described in the form (13), for the one-dimensional case, is independent of the form in which the boundary conditions on the heated surface are given. Connecting it to a concrete type of such conditions is carried out by redefining the functions $\tilde{\theta}(k, z)$ in terms of the external-influence function. We shall perform this operation for boundary conditions of the first kind, when the external-influence function is the temperature of the heated surface:

$$\rho = 1, \quad t(\rho = 1, \tau) = y_1(\tau). \quad (15a)$$

When we pass to discrete values of the external-influence function defined for discrete values of the argument, $F_{0n} = nf$, we have

$$y_1[n] = t[1, n] = t[1, F_{0n}] = t(1, \tau_n). \quad (15b)$$

Successively applying to the last relation first the Kirchhoff transform and then the Z-transform, we obtain

$$\vartheta_1[n] = \vartheta[1, n] = \Theta[1, n] - \Theta_1; \quad \Theta[1, n] = K\{t[1, n]\}; \quad (15c)$$

$$Y_1(z) = Z\{\vartheta[1, n]\} = \tilde{\vartheta}(1, z). \quad (15d)$$

Using the expression (13), written for the case $\rho = 1$, we find

$$\tilde{\vartheta}(k, z) = Y_1(z) \left[\sum_{v=0}^m A_v^1 z^{-v} \right]^{-1}; \quad A_v^1 = A_v(1, f). \quad (16)$$

We make the transition from the mapping functions to the original functions (i.e., we carry out the inverse Z-transform) in the expressions (13) and (16). As a result, we obtain

$$\vartheta[k, n] = \vartheta[1, n] (A_0^1)^{-1} - \sum_{v=1}^m \gamma_v^1 \vartheta[k, n-v]; \quad \vartheta[1, n] = \vartheta_1[n]; \quad (17a)$$

$$\vartheta[\rho, n] = \sum_{v=0}^m A_v(\rho, f) \vartheta[k, n-v]; \quad (17b)$$

$$\vartheta[\rho, n] = \vartheta[1, n] \gamma_0^1(\rho) + \sum_{v=1}^m \omega_v^1(\rho) \vartheta[k, n-v],$$

where $\gamma_0^1 = A_v^1/A_0^1$, $v \geq 1$; $\gamma_0^1(\rho) = A_0(\rho, f)/A_0^1$; $\omega_v^1(\rho) = A_v(\rho) - A_0(\rho) \gamma_v^1$, which define the functions $\vartheta[\rho, n]$ for boundary conditions of the first kind.

In the case of a uniform initial temperature field, the values $\vartheta[k, j]$ for $j \leq 0$ are identically equal to zero, which must be taken into account in using the expressions (17) at the first steps of the calculation, i.e., for $n \leq m$.

The relations obtained can also be used for nonzero initial conditions. For this, we must determine the values of the function $\vartheta[k, j]$ for $j=0, -1, -2, \dots, -m$, corresponding to the given initial temperature field of the body.

$$\tau = 0, \quad t(\rho, 0) = t_i(\rho).$$

To do this, we select a number of points along the thickness of the body with coordinates ρ_i , including the point $\rho = k$, we find

$$\Theta_i(\rho_i) = \int_0^{t_i(\rho_i)} \bar{\lambda}(t) dt; \quad \vartheta[\rho_i, 0] = \Theta_i(\rho_i) - \Theta_i(k).$$

Obviously, $\vartheta[k, 0] = 0$. The remaining values of the function $\vartheta[k, j]$ (for $j = -1, \dots, -m$) are determined by solving the system of equations arising from the expression (17b):

$$\vartheta_i^1(\rho_i) = \sum_{v=1}^m A_v(\rho_i, f) \vartheta[k, -v], \quad i = 1, 2, 3, \dots, q.$$

The total number q of the points must be not less than m (if we exclude the point $\rho = k$). When $q > m$, the system is solved by the method of least squares.

Thus, from the expressions (17), we can calculate the value of the function $\vartheta[\rho, n]$ at the current step of the calculation.

The determination of the actual value of the desired temperature involves an inverse Kirchhoff transform operation:

$$\Theta[\rho, n] = \vartheta[\rho, n] + \Theta_i; t[\rho, n] = K^{-1}\{\Theta[\rho, n]\}, \quad (18)$$

carried out by formulas that can be obtained with the aid of relations (4a), (4b), if we know the expression $\bar{\lambda}(t)$ defining the thermal conductivity as a function of temperature. For example, for the linear function $\bar{\lambda}(t) = 1 + \bar{\lambda}_t t$, which is of great practical interest, we have

$$\Theta(\rho, \tau) = K\{t(\rho, \tau)\} = t(\rho, \tau)[1 + \bar{\lambda}_t t(\rho, \tau)]; \quad (19a)$$

$$t(\rho, \tau) = K^{-1}\{\Theta(\rho, \tau)\} = \bar{\lambda}_t^{-1} \{ [1 + 2\bar{\lambda}_t \Theta(\rho, \tau)]^{1/2} - 1 \}. \quad (19b)$$

The function $\bar{\lambda}(t)$ can also be given in tabular form; then for the direct and inverse Kirchhoff transform we must set up an appropriate table with the aid of the expressions (4).

When we have found the value of $t[\rho, n]$ at a given step of the calculation, we must also determine the real ("local" time $\tau_n(\rho)$ to which the temperature value obtained corresponds. To do this, first, on the basis of the known functions defining how the thermal diffusivity varies with temperature, we determine the value $\bar{a}_n(\rho)$, corresponding to $t[\rho, n]$, and then the value $t[\rho, n]$, which is the average over the temperature interval from $t[\rho, n-1]$ to $t[\rho, n]$, for example by the formula $\bar{a}_n^{av}(\rho)$

$$\bar{a}_n^{av}(\rho) = \frac{1}{2} [\bar{a}_n(\rho) + \bar{a}_{n-1}(\rho)],$$

after which we use the expressions (9) and (10).

For practical use of the calculation formulas obtained above, we must determine the order of the transition from the values of the external-influence function $y_I(\tau)$ to the discrete value $y_I[n] = y_I[F_0 n] = y_I[\tau_n]$. The difficulties involved in this transition are due to the fact that for a constant value of the regular argument step $f = \Delta F_0$ the corresponding step in real time, $\Delta\tau_n(1) = \Delta\tau_0 / \bar{a}_n^{av}(1)$, is a variable quantity and depends on the temperature of the heated surface, $t(1, \tau_n)$, whose value (as a function of the external influence) is not known in advance. In those cases in which we can, without notably lessening the accuracy of the calculation, take $\bar{a}_n^{av}(1) = \bar{a}_{n-1}(1)$, i.e., determine the value of the thermal conductivity from the value of the temperature at the preceding calculation step, this problem is eliminated. In all other cases, it is indispensable to use iterations whose convergence can be accelerated by using a method characteristic of the numerical technique of this approach, connected with extrapolation. When we use a first-order extrapolator, we obtain [1] the first approximation $\vartheta^{(1)}[1, n] = 2\vartheta[1, n-1] - \vartheta[1, n-2]$.

The test of the calculations are performed in the following sequence: $\vartheta^{(1)}[1, n] \rightarrow \Theta^{(1)}[1, n] = \vartheta^{(1)}[1, n] + \Theta_i \rightarrow t^{(1)}[1, n] = K^{-1}\{\Theta^{(1)}[1, n]\}$. The value obtained (the prognosis) for the temperature of the heated surface is taken as its first approximation, which can be used for determining (in the first approximation) the value of the desired instant of real time $\tau_n^{(1)}$:

$$t^{(1)}[1, n] \rightarrow \bar{a}_n^{(1)}(1) \rightarrow \bar{a}_n^{av} = \frac{1}{2} [\bar{a}_n^{(1)}(1) + \bar{a}_{n-1}^{(1)}(1)] \rightarrow \Delta\tau_n^{(1)} = \frac{\Delta\tau_0}{\bar{a}_n^{av}(1)} \rightarrow \tau_n^{(1)}(1).$$

This, in turn, enables us to determine the corresponding value of the external-influence function. It must be taken into account that when we use numerical computing techniques, the input information, including the current values of the external-influences in the monitoring or control of processes taking place in real time, is usually fixed with a constant step $\Delta\tau_c$. We assume that $\Delta\tau_c < \Delta\tau_n$ and in the memory of the computing system we store the values of the external-influence function for at least the last two instants of time: $\tau_l = l\Delta\tau_c$ and $\tau_{l-1} = (l-1)\Delta\tau_c$. When we reach the condition $\tau_{l-1} \leq \tau_n^{(1)} \leq \tau_l$, the value of the external-influence function is found by interpolation:

$$t^{(2)}[1, n] = y^{(1)}(\tau_n) = y(\tau_{l-1}) \frac{\tau_l - \tau_n^{(1)}}{\Delta\tau_c} + y(\tau_l) \left[1 - \frac{\tau_l - \tau_n^{(1)}}{\Delta\tau_c} \right].$$

After obtaining the second approximation for the temperature of the heated surface, we can, where necessary, extend the iterative process in order to obtain more exact values for $\tau_n(1)$ and $y(\tau_n)$. After obtaining the more exact values for $y(\tau_n)$, we carry out the computation

procedures for determining the desired temperature at the given point of the body with coordinate ρ , successively applying the relations (15c), (17a), (17b), and (18).

The method described above for a problem with boundary conditions of the first kind can, in the case of a one-dimensional temperature field, be carried over without any special difficulty to problems with other boundary conditions. This can be done most simply by reducing problems with boundary conditions of the second and third kinds to the problem under consideration [3, 4]. The method can also be applied to the solution of some two-dimensional problems with boundary conditions of the first and second kinds. For example, in the case of boundary conditions of the first kind, for the same temperature over the entire heated surface, for an axially symmetric body of arbitrary shape, all of the above expressions are valid if the functions $P_v(\rho)$ of one geometric coordinate ρ are replaced by the corresponding functions $P_v(\rho, u)$ of two coordinates. The fields of values of the latter in the case of a body of arbitrary shape cannot be determined analytically; they are obtained by numerically solving a recurrent system of Poisson equations [5]:

$$\nabla^2 P_v(\rho, u) = P_v(\rho, u), \quad v \geq 1; \quad \nabla^2 P_0(\rho, u) = 0$$

with the boundary conditions

$$\frac{\partial P_v}{\partial N}(\Gamma^{**}) = 0, \quad v \geq 0; \quad P_0(\Gamma^*) = 1, 0; \quad P_v(\Gamma^*) = 0, \quad v \geq 1.$$

The solutions of nonlinear heat-conduction problems obtained by means of the Z-transform retain the fundamental quality that is characteristic of the previously obtained solutions for constant transfer coefficients: the possibility of determining the temperatures at any points — for example, at characteristic points — of the body without calculating the entire temperature field. Accordingly, we can construct with the aid of these a set of fairly economical algorithms for monitoring and controlling the thermal state of various industrial objects. In particular, they have been used by the author in constructing algorithms to control processes for the heating of high-powered steam turbines.

NOTATION

$t(x, \tau)$, $t(u, \tau)$, $t(\rho, \tau)$, temperature; t_{init} , initial temperature; $r, x; \rho = r/R_{ins}$, $u = x/R_h$, geometric coordinates; $k = R_{ins}/R_h$; R_{ins} , R_{ins} , radii of the heated and insulated surfaces, respectively; R_{inn} , R_{out} , radii of the inner and outer surfaces, respectively; τ , time; $c(t)$, specific heat; $\lambda(t)$, thermal conductivity; $\gamma(t)$, density; $a(t) = \lambda(t)/(c(t)\gamma(t))$, thermal diffusivity; $\bar{\lambda}(t) = \lambda(t)/\lambda_0$; $\bar{a}(t) = a(t)/a_0$; λ_0 , a_0 , values of $\lambda(t)$ and $a(t)$, respectively, when $t = 0$; $a_n^{av}(\rho)$, average value of the thermal diffusivity in the temperature interval from $t(\rho, \tau_n)$ to $t(\rho, \tau_{n-1})$; Γ^* , Γ^{**} , heated and insulated surfaces of the two-dimensional body, respectively; $y(\tau)$, $y(\tau_n) = y[Fo_n] = y[n]$, external-influence function on the body; i, v, j, m, n, l, q , intergers; Fo , "model" time (Fourier number); $f = \Delta Fo = a_0 \Delta \tau_0 / (R_1 - R_0)^2$; $\Delta \tau_0$, constant; K and K^{-1} , operators of the direct and inverse Kirchhoff transform (substitution); $\Theta(\rho, \tau)$, Θ_i , Kirchhoff mappings for $t(\rho, \tau)$, t_i respectively; $\Phi(\rho, z)$ and $Y(z)$, mappings of the functions $\Phi[\rho, n]$ and $y[n]$, respectively, obtained by means of the Z-transform; z , parameter of the Z-transform (discrete Laplace transform).

LITERATURE CITED

1. V. A. Bessekerskii, Automatic Digital Systems [in Russian], Nauka, Moscow (1976).
2. V. L. Pokhoriler, N. N. Galimulin, and A. I. Shklyar, "Development of a model for the heating of the rotor of a high-powered steam turbine for the ASU TP energy units," Izv. Vyssh. Uchebn. Zaved., Energetika, No. 12, 46-50 (1980).
3. V. L. Pokhoriler, "Solution of problems in heat conduction by the method of the discrete Z-transform," Inzh.-Fiz. Zh., 39, No. 5, 909-915 (1980).
4. V. L. Pokhoriler, A. I. Shlyar, and N. N. Galimulin, "Solution of problems in heat conduction for cylindrical bodies by the method of the discrete Z-transform," Thermophysics of Nuclear Power Installations, first edition [in Russian], Interuniversity Collection, Sverdlovsk: Kirov UPI (1982), pp. 121-127.
5. V. L. Pokhoriler, "Solution of two-dimensional heat-conduction problems by the Z-transform method," Inzh.-Fiz. Zh., 44, No. 5, 818-824 (1983).
6. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).

7. Yu. M. Matsevityi, *Electrical Modeling of Nonlinear Problems in Technical Thermophysics* [in Russian], Naukova Dumka, Kiev (1977).
8. V. Cappelini, A. Constantinidis, and P. Emiliani, *Digital Filters and Their Application* [Russian translation], Energoatomizdat, Moscow (1983).
9. V. L. Pokhoriler, "Using the solution of an inverse heat-conduction problem for calculating the heat-exchange coefficient from the experimentally measured temperatures of the interior points of a body," *Inzh.-Fiz. Zh.*, 23, No. 5, 879-883 (1972).

DETERMINING THE MEAN-SQUARE ERROR AND DISCRETIZATION STEP OF THE
INITIAL DATA OF AN INVERSE PROBLEM IN A SINGLE REALIZATION

A. I. Maiorov and L. A. Rudometkin

UDC 536.24.02

A method is developed allowing the approximate values of the mean square error and optimal discretization step of the initial data to be found from a single realization of a random process.

In solving inverse problems by means of information on the mean square error σ of the initial data, the accuracy of the results obtained depends on the accuracy in determining σ . The economy and accuracy of computer calculations depends largely on the number of discretization points of the initial inverse-problem data.

To determine the optimal discretization step H_{opt} of a random process $T(\tau)$ consisting of a useful signal and an arbitrarily distributed perturbation, it is assumed that the greatest frequency $T(\tau)$ is finite and $T(\tau)$ is specified by the division T_i , $i = 1, \dots, N$, in sufficient detail (no less than three points must cover each halfperiod of the characteristic variations). Using a cubic spline $S_{\Delta}(\tau, T_i)$ interpolating the values of T_i , the characteristic frequency f_{max} of high-frequency oscillations of the function $T(\tau)$ with respect to the number of points N^* of sign change of the second derivative $S''_{\Delta}(\tau, T_i)$ on the given segment $[0, \tau_{\text{max}}]$ is found

$$f_{\text{max}} = \frac{N^*}{2\tau_{\text{max}}}.$$

In accordance with the Kotelnikov and Zheleznov discretization theorem — see [1], for example — the division $S_{\Delta}(\tau, T_i)$ is made with a step equal to half the characteristic period of the high-frequency oscillations, that is, with

$$H_{\text{opt}} = \frac{1}{2f_{\text{max}}}.$$

This step is very close to the maximum possible value at which all the information on the useful signal and the error of the initial data $T(\tau)$ is retained. To determine the mean square error σ of the initial data $T(\tau)$, the squares of the deviations of $S_{\Delta}(\tau, T_i)$ at each internal point of the chosen optimal grid division from the straight lines passing through two adjacent corners are averaged. This leads to the value

$$\delta^2 = \frac{1}{N^* - 1} \sum_{i=2}^{N^*} \left[T_i^* - \frac{1}{2} (T_{i+1}^* - T_{i-1}^*) \right]^2,$$

where T_i^* , $i = 1, \dots, N^*$, are the corner values of the optimal grid division. To determine the difference of δ from σ , the error of the initial data is specified using a harmonic function of the form

$$\varepsilon(\tau) = a \sin(2\pi f_{\text{max}} \tau).$$

Translated from *Inzhenerno-Fizicheskii Zhurnal*, Vol. 51, No. 1, pp. 150-152, July, 1986.
Original article submitted May 17, 1985.